On spectral Actions

Bruno Iochum

Mathematisches Forschungsinstitut Oberwolfach

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Mini-Workshop: Dirac Operators in Differential and Noncommutative Geometry

Organised by
Christian Bär (Potsdam)
Andrzej Sitarz (Krakow)

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Abstract.
This mini-workshop brought together mathematicians and physicists working either on classical or on noncommutative differential geometry. Our aim was to show current interests, methods and results within each group and to open the possibility for interaction between the two groups. The first three days were devoted to expository presentations. The remaining two days were devoted to talks on advanced current research problems and results.

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During the talk, based on collaborations with V. Gayral, C. Levy, A. Sitarz and D. Vassilevich, different aspects of the Connes–Lott action and Chamseddine–Connes spectral action for spectral triples are developed. New results concerning the second action for the noncommutative torus and the triple associated to $SU_q(2)$ are presented.

Since the beginning of noncommutative geometry [6, 15], the notion of action, essential in physics, has had two main evolutions. The first proposed by Connes and Lott [8] was based on the following formula, associated to a spectra triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension d:

$$YM(\alpha) = \inf_\eta \{ \operatorname{Tr}_{\text{Dixmier}}\big(F^*F|\mathcal{D}|^{-d}\big) \,:\, \alpha = \pi(\eta) \}.$$

Here, π is the representation of $\mathcal{A}$ on $\mathcal{H}$, η is a one-form and $F = \delta\eta + \eta^2$ is its curvature where δ is a formal derivation on $\mathcal{A}$ represented on $\mathcal{H}$ by $\delta(a) \to [\mathcal{D}, \pi(a)]$, $a \in \mathcal{A}$. In the commutative case, $YM(\alpha)$ is the usual Yang–Mills action. The point is that, to get a graded differential algebra after the representation, one has to divide by an ideal, so for explicit computations, one has to control that "junk". Moreover the use of the Dixmier trace implies that only the leading term of the poles is concerned. The second step was the appearance of the spectral action proposed by Chamseddine and Connes [3]

$$\mathcal{S}(\mathcal{D}, \Lambda, \Phi) := \mathrm{Tr}\big(\Phi(|\mathcal{D}|/\Lambda)\big),$$

where Φ is a positive function and Λ is a scale used for a cut-off purpose. The idea is that if Φ looks like the step function, then this counts the number of eigenvalues of $|\mathcal{D}|$ less than Λ. Its main interest is of course that it depends only of the spectrum of the Dirac operator $\mathcal{D}$, an interesting fact since there exist isometric non isospectral manifolds. It is of course gauge invariant.
Since it is based on a heat kernel approach, it is important to get precise constraints on Φ for applying right holomorphic extensions. It is also possible to a distributional approach like in [10]. However, if one considers the circle of ideas stemming from the heat kernel approach, the zeta function approach or the Dixmier-trace approach, one may note few differences on the hypothesis sufficient to use one of these entrances.
Connes introduced in [7] the notion of spectrum dimension $Sd(\mathcal{A}, \mathcal{D}, \mathcal{H})$ of a spectral triple by the singularities of the function $\zeta_b(s) := \mathrm{Tr}(b|\mathcal{D}|^{-s}) + \mathrm{Tr}(pb)$ when b is in the set of pseudodifferential operators generated by $\mathcal{A}$, $\delta^n(\mathcal{A})$, $\delta^n([\mathcal{D}, \mathcal{A}])$ for all $n \in \mathbb{N}$ or powers of $|\mathcal{D}|$, where $\delta(.) = [|\mathcal{D}|, .]$ and p is the projection on the kernel of b. Few remarks:
- When M is a Riemannian compact spin manifold, then

$$Sd(\mathbb{C}^\infty(M), \mathcal{L}^2(M, \text{Spinor bundle}), \text{Dirac operator}) \subset \{\, 1, \cdots n \,\}.$$

- It makes sense to define $⨍ b := \mathrm{Res}_{s=0}\, \zeta_b(s)$ and $⨍$ is a trace on pseudodifferential operators.

- A way to check that an operator is Dixmier traceable is the following [1]: If T is a positive bounded operator on $\mathcal{H}$ such that T^s is trace-class for $s > 1$ and $l = \lim_{0<\epsilon\to 0} \epsilon \mathrm{Tr}(T^{1+\epsilon})$ exists, then T is measurable and $\mathrm{Tr}_{\mathrm{Dixmier}}(T) = l$.

- The action is $\mathcal{S}(\mathcal{D}, \Lambda, \Phi) \sim \sum_{0\leq k\in Sd} \Phi_{n-k} ⨍ |\mathcal{D}|^{-k}\, \Lambda^k + \Phi(0)\zeta_{\mathcal{D}}(0) + o(1)$.

Example 1: $\mathcal{A} = \mathcal{A}_\Theta$ is the noncommutative n-torus associated to a skew-symmetric deformation matrix Θ with the Hilbert space $\mathcal{H}$ related to the trace τ on $\mathcal{A}_\Theta$ and the Dirac operator based on the natural derivations δ_μ, $\mu \in \{\, 1 \cdots n \,\}$ such that $\delta_\mu U_k = ik_\mu U_k$ where $\mathcal{A}$ is generated by the unitaries U_k satisfying $U_k U_q = e^{ik.\Theta q}\, U_q U_k$. The natural reality operator J is associated to $J_0(a) = a^*$.
For a selfadjoint one-form $A = \sum_i a_i[\mathcal{D}, b_i]$, a_i, $b_i \in \pi(\mathcal{A})$ (π is the representation on $\mathcal{H}$), the fluctuated Dirac operator is $\mathcal{D}_A := \mathcal{D} + A + JAJ^{-1} = -i\big(\delta_\mu + L(A_\mu) - R(A_\mu)\big) \otimes \gamma^\mu$, with $A_\mu^* = -A_\mu \in \mathcal{A}$ and the γ matrices are selfadjoint.

Using the result $\zeta_{\mathcal{D}_A}(0) - \zeta_D(0) = \sum_{k=1}^n \frac{(-1)^k}{k} ⨍ (A\mathcal{D})^k$ obtained in [4] and [11], we get by a heat kernel expansion that the spectral action in dimension $n = 4$ is

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi^2\Phi_0\Lambda^4 - \tfrac{4\pi^2}{3}\Phi(0)\tau(F_{\mu\nu}F^{\mu\nu}) + o(1)$$

where $\Phi_0 := \frac{1}{2}\int_0^\infty x\,\Phi(x)\,dx$ and $F_{\mu\nu} = \delta_\mu(A_\nu) - \delta_\nu(A_\mu) + [A_\mu, A_\nu]$. This result [16] is the same as for the classical 4-torus and was obtained in [14] but as a by-product of more general results under the assumption that Θ satisfies a Diophantine condition.

It is interesting to quote that some of the techniques has been extended to non-compact manifolds [13].

Example 2: The spectral triple introduced in [9, 17] is based on the quantum $SU(2)$: Let $\mathcal{A} = \mathcal{A}(SU_q(2))$ be the $*$-algebra generated polynomially by a and b, subject to the following commutation rules:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1.$$

with $0 < q < 1$. The spinorial Hilbert space $\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ has an orthonormal basis consisting of vectors $|j\mu n\!\uparrow\rangle$ for $j = 0, \frac{1}{2}, 1, \dots,\ \mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$; together with $|j\mu n\!\downarrow\rangle$ for $j = \frac{1}{2}, 1, \dots,\ \mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$ (here $x^\pm := x \pm \frac{1}{2}$). Using the vector notation $|j\mu n\rangle\!\rangle := (|j\mu n\!\uparrow\rangle, |j\mu n\!\downarrow\rangle)$, with the convention that the lower component is zero when $n = \pm(j+\frac{1}{2})$ or $j = 0$, the Dirac operator is chosen the same as in the classical case of a 3-sphere:

$$\mathcal{D}|j\mu n\rangle\!\rangle = \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle\!\rangle.$$

It is sufficient to use the approximate spinorial representation $\underline{\pi}$ of $SU_q(2)$ presented in [17, 9] since all disregarded terms are trace-class and do not influence residue calculations. Moreover, we may replace $\mathcal{D}$ by $|\mathcal{D}|$.

Here $Sd = \{\,1, 2, 3\,\}$ so the behavior is totally different from Example 1. First, there exist non vanishing tadpoles $\Psi_1(A) := ⨍ A\mathcal{D}^{-1}$. In fact, Ψ_1 is a cyclic cocycle with a nontrivial pairing with the generator of K_1 group. One also get
$⨍ A_1\mathcal{D}^{-1}A_2\mathcal{D}^{-1}A_3\mathcal{D}^{-1} = ⨍ A_1A_2A_3\mathcal{D}^{-3}$ or

$$⨍ a_0[\mathcal{D},a_1]\mathcal{D}^{-1}a_2[\mathcal{D},a_3]\mathcal{D}^{-1} = ⨍ a_0[\mathcal{D},a_1][\mathcal{D},a_2[\mathcal{D},a_3]\mathcal{D}^{-3} + ⨍ a_0[\mathcal{D},a_1]a_2\mathcal{D}^{-1}[\mathcal{D},a_3]\mathcal{D}^{-1}.$$

More generally, the inner fluctuations gives for the scale-invariant part of the spectral action given by a universal 3-form $A = a_0da_1da_2da_3$:

$$\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \tfrac{1}{2}\int_{\Psi_3}(AdA + \tfrac{2}{3}A^3) + \int_{\Psi_2} A^2 - \int_{\Phi_2} A^2 - \Psi_1(A)$$

with $\Psi_3(a_0,a_1,a_3) = ⨍ a_0[\mathcal{D},a_1][\mathcal{D},a_2][\mathcal{D},a_3]\mathcal{D}^{-1}$, $\Psi_2 = ⨍ a_0[\mathcal{D},a_1][\mathcal{D},[\mathcal{D},a_2]]\mathcal{D}^{-2}$, and $\Phi_2(a_0,a_1,a_2) = ⨍ a_0[\mathcal{D},a_1][\mathcal{D},a_2]\mathcal{D}^{-2}$. The first term is of course the Chern-Simons term.

Example 3: Different applications of the spectral action in particle physics can be found since a long time in [2, 3]. A new extension has been made recently in [5] for a case probably appropriate to Lorentzian geometry.

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